

Sensitive Light: Geometry, Chaos, and Information Recovery in Null Transport

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Abstract. How sensitively does light propagation depend on initial conditions, how far ahead can it be predicted, and how much information about the source survives in partial observations? This paper measures all three across geometries and media, keeping four regimes distinct: stable, unstable but non-chaotic, chaotic, and predictably reconstructable. For unstable photon orbits the organizing quantity is the ratio of the coordinate-time Lyapunov exponent to the orbital frequency, λ_t/Ω_c : it equals 1 for Schwarzschild and for the Ellis wormhole throat (whose exponent we compute in closed form, $\lambda_t = 1/a$), collapses to 0.134 for near-extremal prograde Kerr, and equals $\sqrt{d-3}$ for Tangherlini black holes (proved here). Scattering measurements confirm the framework: the winding-number divergence near the critical impact parameter fits the predicted coefficient $1/(2\pi)$ to 0.3%, and the arrival-time sensitivity per winding grows with slope 6.323 against the analytic 2π . The Majumdar–Papapetrou di-hole, the one genuinely chaotic spacetime computed, has a fractal exit-basin boundary confined to a thin set (uncertainty exponent $\alpha = 0.669$, boundary dimension 0.33 on a one-dimensional cut, first such measurement for this system), no long-lived rays in $\sim 51,000$ integrated rays (so no escape rate is measurable at these parameters), and outcome statistics that remain 99.3% recoverable under coarse-graining at resolution 10^{-2} . A double plasma lens is, by contrast, regular ($\alpha = 1.01$) and remains regular across a strong-coupling scan (adjudicated $\alpha = 0.95$ – 1.09 , with backscattering but no trapping), and its angular focus desynchronizes arrival times by a factor of 54. On information recovery the paper separates conditional from expected Fisher information: near-critical rays amplify parameter derivatives as $e^{\lambda T}$ while their initial-condition window shrinks as $e^{-\lambda T}$, and we state the resulting chaos–information tradeoff as a conjecture with three named cutoffs (wave optics, finite source size, dwell time), against the control case of an integrable point lens whose Cramér–Rao bound saturates at the caustic. An explicit accounting separates established results, synthesis, and candidate new results, and names the open problems (a Fisher theory for chaotic observables; a Melnikov criterion for chaos onset) that the measurements leave standing.

Keywords: null geodesics, Lyapunov exponents, chaotic scattering, black holes, wormholes, gravitational lensing, Fisher information, predictability

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1 Introduction

A companion study (Kilambi Research, 2026) asked which geometries bring a broad family of light rays from one point to another, and with what synchronization of arrival times. This paper asks the orthogonal question: given a geometry, how sensitive is light propagation to initial conditions, how far ahead can it be predicted, and how much information about the source survives in partial observations? The three questions are linked by a single mechanism. The optical tidal matrix that focuses a congruence also amplifies deviations between neighbouring rays, so the geometries that focus most strongly are candidates for the strongest sensitivity, and the same amplification that destroys long-range predictability also magnifies the response of observables to source displacements. Whether that magnification is a resource or a liability for reconstruction depends on quantitative details that this paper measures.

The paper distinguishes four regimes of null propagation (Section 2.3): stable, unstable but non-chaotic, chaotic, and predictably reconstructable. The distinction between the second and third is the paper’s conceptual spine. A Schwarzschild black hole has an unstable photon sphere with Lyapunov exponent $\lambda = 1/(3\sqrt{3}M)$, yet its photon dynamics is integrable and its scattering functions are smooth except at one critical impact parameter. A two-body geometry such as the Majumdar–Papapetrou di-hole is non-integrable: its scattering functions develop fractal structure, its outcome basins have fractal boundaries, and its trapped rays form a chaotic repeller. Claims in the literature that “black hole photon motion is chaotic” are usually statements of the first kind wearing the vocabulary of the second. Section 2 fixes definitions precise enough to separate them, and the numerical sections then assign each studied geometry to its regime with numbers, not labels.

What this paper contributes:

1. A verification of photon-sphere Lyapunov exponents by two independent methods (deviation integration against the closed form of Cardoso et al., 2009) for Schwarzschild, a spin-resolved table for Kerr equatorial photon circles, and a measured winding-number divergence $n(b) \simeq -(\Omega_c/2\pi\lambda) \ln |b - b_c|$ whose fitted coefficient tests the invariant Ω_c/λ directly (Section 3).
2. A quantitative chaotic-scattering study of the Majumdar–Papapetrou di-hole: fractal scattering function, the first uncertainty exponent and boundary dimension for its exit basins, an attempted escape-rate measurement that returns a bounded null (no long-lived rays in $\sim 51,000$), and a scale-by-scale measure of how much outcome information survives coarse-graining of the initial data (Section 4).
3. Closed-form Lyapunov results for two further classes: the Ellis wormhole throat orbit, $\lambda_t = 1/a$ with $\lambda/\Omega_c = 1$; and Tangherlini black holes in spacetime dimension d , for which we prove $\lambda/\Omega_c = \sqrt{d-3}$ (Proposition A.1), verified numerically for $d = 5$ (Section 5).
4. A ray-optics study of single and double plasma lenses in an engineered medium: at moderate coupling the double lens remains regular (uncertainty exponent $\alpha = 1.01$, a bounded null against the di-hole’s 0.669), and its single-lens angular focus desynchronizes arrival times by a factor of 54, separating angular focusing from temporal synchronization (Section 5).
5. A Fisher-information analysis of source localization that separates conditional from expected information: near-critical rays carry parameter derivatives growing as $e^{\lambda T}$, while the initial-condition window producing them shrinks as $e^{-\lambda T}$. We state the resulting tradeoff as a conjecture with numerical support and identify the conditions under which chaotic sensitivity is a net resource for reconstruction (Sections 3 and 6).

Status conventions follow the companion paper: THEOREM and LEMMA (proved, here or cited), COMPUTATION (computed here, method stated), CONJECTURE (open, labelled). Quantitative claims carry the evidence vocabulary Observed / Reconstructed / Modelled / Assumed with grades A/B/C. Signature $(-, +, +, +)$ and $G = c = 1$ throughout; black hole masses are set to unity in numerical work unless stated. Section 2 builds the framework. Section 3 treats single black holes, Section 4 the di-hole, Section 5 wormholes, higher dimensions, and plasma media. Section 6 assembles the cross-geometry comparison, states the chaos–information tradeoff, and gives an explicit accounting of which results are established, which are synthesis, and which are new. Section 7 states falsifiers and limitations, and Section 8 concludes.

2 Framework: deviation, Lyapunov exponents, and information

2.1 Null congruences and the deviation equation

Let k^a be the tangent to an affinely parametrized null geodesic and let ξ^a be a connecting vector to a neighbouring geodesic of the same congruence, Lie-dragged along k^a and projected onto the two-dimensional screen space orthogonal to k^a and to an auxiliary null vector. In screen components ξ_i , $i = 1, 2$, the geodesic deviation equation is

$$\frac{d^2 \xi_i}{d\lambda^2} = -\mathcal{K}_{ij} \xi_j, \quad \mathcal{K}_{ij} = R_{acbd} k^c k^d e^a_i e^b_j, \quad (1)$$

where e^a_i are screen basis vectors and $\mathcal{K}_{ij}$ is the optical tidal matrix. Its trace drives focusing through the null Raychaudhuri equation; its eigenvalues, integrated along the ray, drive sensitivity to initial conditions. Both phenomena studied in this paper, focusing and chaos, are therefore properties of one object, $\mathcal{K}_{ij}$, read in two ways: the trace accumulated along a single ray, and the growth of the fundamental matrix of (1) across a congruence.

2.2 Lyapunov exponents for null congruences

Definition 2.1 (Finite-time Lyapunov exponent). For a ray segment of duration T (in a stated parameter: affine parameter, coordinate time t , or optical path length), and an initial deviation of norm δ_0 in the screen space, the finite-time Lyapunov exponent is

$$\lambda_T = \frac{1}{T} \ln \frac{\|\xi(T)\|}{\|\xi(0)\|}, \quad (2)$$

with $\|\cdot\|$ the screen-space norm. The Lyapunov exponent $\lambda = \lim_{T \rightarrow \infty} \lambda_T$ is taken along the ray or orbit of interest, with the parameter stated in every use. For scattering problems T is finite by construction and λ_T , not λ , is the operative quantity. Existence of the limiting exponents is the content of the multiplicative ergodic theorem (Oseledets, 1968); their positive sum is the Kolmogorov–Sinai entropy under Pesin’s identity (Pesin, 1977; Eckmann and Ruelle, 1985). Finite-time exponents of this kind are the standard diagnostic of Lagrangian transport (Shadden et al., 2005).

For an unstable circular photon orbit in a static spherical metric $ds^2 = -f(r) dt^2 + f(r)^{-1} dr^2 + r^2 d\Omega^2$, the null radial equation is $\dot{r}^2 + V(r) = E^2$ with $V(r) = f(r)L^2/r^2$ (dot: affine parameter). A

circular orbit sits at an extremum r_c of V , and linearizing the radial equation about r_c gives the coordinate-time Lyapunov exponent

$$\lambda_t^2 = \frac{f(r_c)}{2r_c^2} \left[2f(r_c) - r_c^2 f''(r_c) \right], \quad \Omega_c^2 = \frac{f(r_c)}{r_c^2}, \quad (3)$$

where Ω_c is the orbital frequency (Cardoso et al., 2009). Equation (3) is the workhorse of Sections 3 and 5: every photon-sphere exponent reported there is computed from it and cross-checked against direct deviation integration.

A parametrization caveat applies to every exponent in this paper. Lyapunov exponents in general relativity depend on the time parameter of the parametrization and are diagnostics, not observables (Cornish and Levin, 1997). We therefore (i) state the parameter with every exponent (coordinate time t of an asymptotic observer unless noted); (ii) emphasize the ratio λ/Ω_c , which is invariant under constant rescalings of t and which governs observable winding counts through (5); and (iii) rest all claims of chaos, as opposed to instability, on parametrization-independent diagnostics: fractal scattering functions, uncertainty exponents, and basin structure (Dettmann et al., 1995; Cornish and Levin, 1997).

2.3 Four regimes of light propagation

The literature often uses “unstable” and “chaotic” interchangeably for photon motion. They are different, and the difference is load-bearing for every claim in this paper. We use the following classification throughout.

Definition 2.2 (Regimes of null propagation). A family of null geodesics in a fixed geometry is:

1. **Stable (regular)** if deviations grow at most polynomially along the family: $\lambda = 0$ and no exponential sensitivity to initial conditions.
2. **Unstable (non-chaotic)** if the geometry possesses isolated unstable periodic null orbits with $\lambda > 0$, so that rays asymptotic to those orbits separate exponentially, but the dynamics is integrable (or effectively integrable) and the set of exponentially sensitive rays has measure zero in initial-condition space. Scattering functions are smooth except at isolated critical values.
3. **Chaotic** if the dynamics is non-integrable with $\lambda > 0$ on a set of initial conditions of positive measure: scattering functions are fractal, outcome basins have fractal boundaries (uncertainty exponent $\alpha < 1$, Section 2.5), and the trapped set is a chaotic repeller with positive Kolmogorov–Sinai entropy.
4. **Predictably reconstructable** (a property of an observation scheme on a geometry, not of the geometry alone) if the map from source and geometry parameters to the recorded observables has a Fisher matrix of full rank with controlled conditioning, so that reconstruction errors are finite and estimable by the Cramér–Rao bound at the stated noise level.

Schwarzschild and Kerr photon motion are unstable in this sense, not chaotic: both are integrable, and exponential sensitivity is confined to rays that asymptote to the photon sphere or spherical photon orbits (Carter, 1968; Teo, 2003). The Majumdar–Papapetrou di-hole and the double plasma lens of Section 4 and Section 5 are chaotic in this sense. Conflating the two regimes produces false claims in both directions: instability alone implies neither fractal scattering nor positive entropy, and chaos does not by itself destroy reconstructability (Section 6).

2.4 Predictability horizons

Given an initial-condition uncertainty δ_0 and a tolerance Δ at which a prediction is counted as lost, the predictability horizon of a ray family with exponent λ (the logarithmic dependence on precision is the classical one (Lorenz, 1963)) is

$$T_{\text{pred}} = \frac{1}{\lambda} \ln \left(\frac{\Delta}{\delta_0} \right). \quad (4)$$

For scattering off an unstable photon orbit this has a concrete reading. A ray with impact parameter b near the critical value b_c dwells near the photon orbit for a time $T(b) \simeq -\lambda^{-1} \ln |b - b_c| + O(1)$, because its initial offset from the stable manifold is $O(|b - b_c|)$ and it leaves when the deviation has grown to $O(1)$. During the dwell it winds $n(b) \simeq \Omega_c T(b)/(2\pi)$ times, so

$$n(b) \simeq -\frac{\Omega_c}{2\pi\lambda} \ln |b - b_c| + O(1), \quad \frac{dT}{db} \simeq \frac{1}{\lambda |b - b_c|}. \quad (5)$$

The ratio Ω_c/λ is thus a measurable invariant of the critical orbit: it fixes the winding-number divergence, and through dT/db it fixes how strongly arrival times respond to source displacements. Section 3 measures it for Schwarzschild and Kerr; Section 5 for the Ellis wormhole and for Tangherlini black holes in dimensions four to seven, where we find $\lambda/\Omega_c = \sqrt{d-3}$ (Proposition A.1).

2.5 Scattering measures: uncertainty exponent, escape rate, basin entropy

For chaotic scattering we use three standard quantitative measures (Ott and Tél, 1993; McDonald et al., 1985; Bleher et al., 1990; Aguirre et al., 2009; Seoane and Sanjuán, 2013; Grebogi et al., 1987; Daza et al., 2016). First, the *uncertainty exponent*: if the outcome of a ray is determined by an initial condition x (here an impact parameter), the fraction $f(\epsilon)$ of points whose outcome is changed by a perturbation of size ϵ scales as $f(\epsilon) \sim \epsilon^\alpha$. In a one-dimensional initial-condition cut the basin boundary has fractal dimension $D = 1 - \alpha$; regular scattering gives $\alpha = 1$, $D = 0$. Second, the *escape rate* γ : the dwell-time distribution of the trapped set decays as $P(T) \sim e^{-\gamma T}$. Third, *basin entropy*: the Shannon entropy of outcome probabilities within boxes of size ϵ , averaged over the initial-condition region, which quantifies outcome uncertainty at finite resolution. The escape-rate relation $\gamma = \bar{\lambda} - h_{\text{KS}}$ connects the escape rate to the mean Lyapunov exponent $\bar{\lambda}$ of the repeller and its Kolmogorov–Sinai entropy h_{KS} (Ott and Tél, 1993). Section 4 attempts the corresponding measurement on the di-hole repeller and reports why it does not execute at accessible parameters: no exponential dwell-time tail is produced.

2.6 Fisher information and the reconstruction problem

Let observables O_μ (image positions, arrival times, flux ratios) depend on parameters θ (source position, geometry parameters), with Gaussian noise of covariance Σ . The Fisher matrix (Fisher, 1925) $F_{ij} = \sum_\mu \Sigma_{\mu\mu}^{-1} (\partial O_\mu / \partial \theta_i) (\partial O_\mu / \partial \theta_j)$ bounds the covariance of any unbiased estimator, $\text{Cov}(\hat{\theta}) \succeq F^{-1}$ (Cramér–Rao). Two consequences organize the information half of the paper. First, by (5), arrival-time observables of near-critical rays carry parameter derivatives that grow like $1/|b - b_c| \sim e^{\lambda T}$: conditional on observing such a ray, source localization sharpens exponentially with dwell time. Second, the measure of initial conditions that produce such rays shrinks at the same exponential rate. Whether chaos and instability help or harm reconstruction is therefore a question about the product of the two, not about either factor separately; we call the product the *expected information density* and return to it in Section 6.

3 Single black holes: instability without chaos

3.1 Schwarzschild: the exponent, measured three ways

For Schwarzschild ($M = 1$), (3) at $r_c = 3$ gives $\lambda_t = \Omega_c = 1/(3\sqrt{3}) = 0.192450$ (Cardoso et al., 2009) (RECONSTRUCTED, A). We verify this value by two further, independent routes, because every later section uses photon-orbit exponents as inputs.

First, direct deviation integration: a null geodesic started at $r = 3 + 10^{-6}$ with its paired ray shows screen-space deviation growth that is exponential over four decades of $|\delta r|$ (Figure 1); the fitted rate is $\lambda_t = 0.192308$, a relative error of 0.074% against the closed form (MODELLED, B). In the affine parameter the same integration gives $\lambda_{\text{aff}} = 0.111029$ against the predicted $1/9 = 0.111111$ for the $L = 1$ normalization: the two exponents differ by exactly $t = 1/\sqrt{3}$ on the orbit, a concrete illustration of the parametrization caveat of Section 2.2. Only λ_t is normalization-independent.

Second, the scattering divergence of Section 2.4: across 140 impact parameters refined to $|b - b_c| = 10^{-4}$, the winding number fits $n(b) = -A \ln |b - b_c| + B$ with $A = 0.15868$ on the escaped side and $A = 0.15958$ on the captured side ($R^2 = 0.999997$ in both), against the predicted $A = \Omega_c/(2\pi\lambda_t) = 1/(2\pi) = 0.159155$: agreement within 0.3% (MODELLED, B). The dwell-time divergence, fitted independently, returns $\lambda_t = 0.192969$ (+0.27%). Three routes to the same exponent (potential curvature, deviation growth, dwell-time divergence) agree at the sub-percent level; the exponent is not the uncertain part of any later argument.

The finite-time Lyapunov exponent over whole trajectories peaks at the grid point nearest b_c with $\lambda_T = 0.11576$ (Figure 2). It sits below λ_t , as it must: most of a scattering trajectory is free flight through weak field, which dilutes the average. The peak-to-baseline ratio is ≈ 1.6 . Whole-trajectory FTLE is therefore a locator of critical structure, not a substitute for the orbit exponent, and we use it only in that role.

3.2 Kerr: spin splits the exponent

Kerr geodesic motion is integrable (Carter, 1968), so Kerr belongs to the unstable, non-chaotic regime. The equatorial circles below are the simplest members of a full periodic-orbit taxonomy that includes the homoclinic separatrix, the standard seed of chaos under perturbation (Levin and Perez-Giz, 2008, 2009; Perez-Giz and Levin, 2009), and exact Kerr null geodesics are available in closed form (Gralla and Lupsasca, 2020b); but its equatorial photon circles show that “the” Lyapunov exponent of a black hole is not one number. Table 1 gives radii, frequencies, and exponents from the radial-potential curvature method, cross-checked at $a = 0.9$ by near-critical scattering: dwell-time and winding log-slopes return $\lambda = 0.14011$ and a winding coefficient of 0.39805, against the potential-method values 0.13608 and $\Omega_c/(2\pi\lambda) = 0.41118$ (deviations of +3.0% and −3.2%; the scattering check uses log-slopes rather than a paired-ray FTLE, a method difference we flag rather than hide).

The physics is in the ratio. A near-extremal prograde photon circle is only weakly unstable: $\lambda_t = 0.0597$ at $a = 0.99$, a third of its Schwarzschild value, while Ω_c more than doubles. The winding coefficient $A = \Omega_c/(2\pi\lambda_t)$ therefore grows to 1.18: each decade of approach to b_c buys more than a full extra turn around the hole. The retrograde circle, by contrast, is nearly spin-blind. Any observational programme that reads instability off photon-ring structure (Gralla and Lupsasca, 2020a; Johnson et al., 2020; Wielgus, 2021) is reading the prograde ratio, and the ratio varies by an

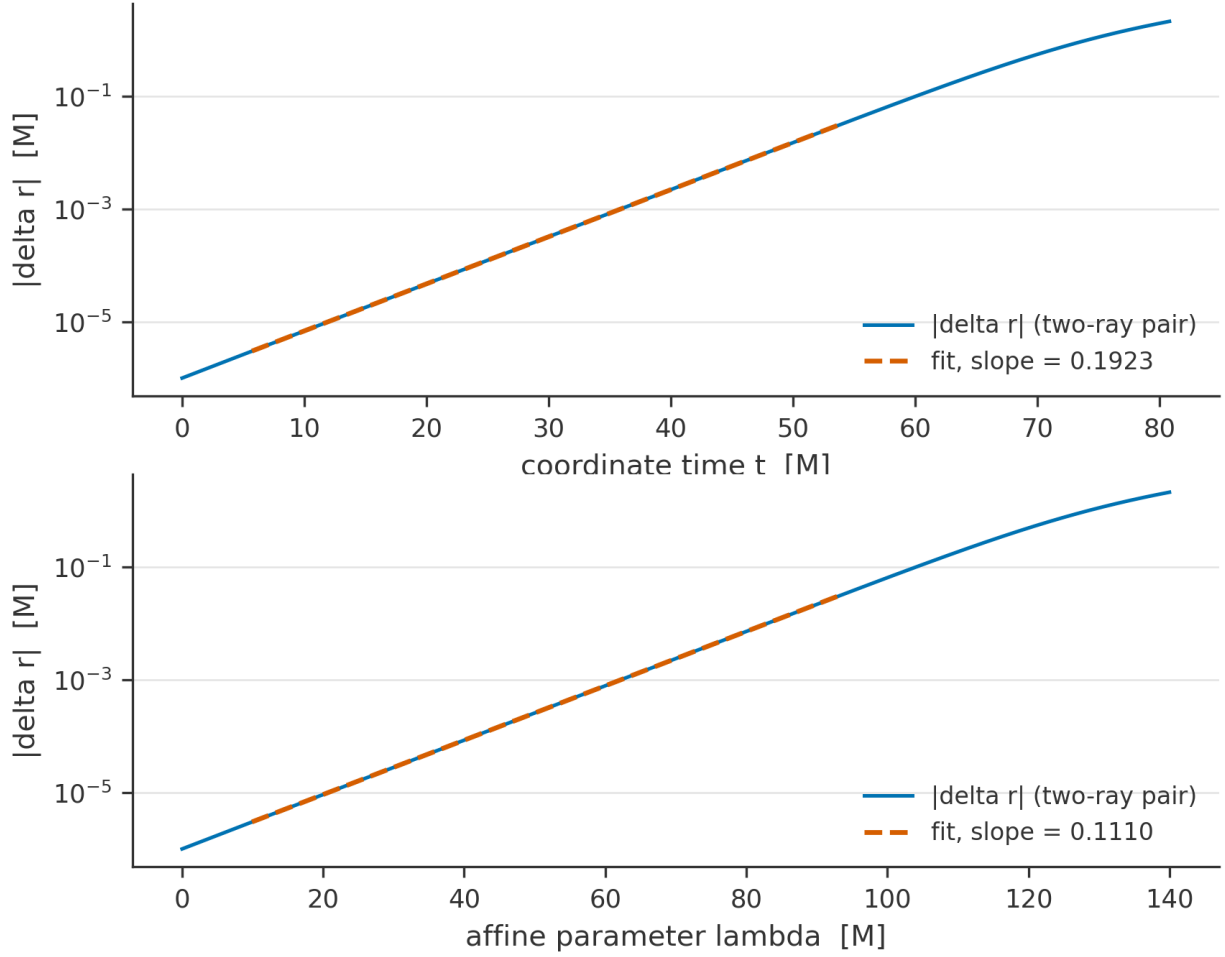


Figure 1. Deviation growth at the Schwarzschild photon sphere. Source: `code/exp_bh.py`, run E1. Note: log scale; fit over $3 \times 10^{-6} < |\delta r| < 3 \times 10^{-2}$.

Table 1. Equatorial circular photon orbits in Kerr, $M = 1$: radius r_c , orbital frequency Ω_c , coordinate-time Lyapunov exponent λ_t . The $a = 0$ row (Schwarzschild) is $r_c = 3$, $\Omega_c = \lambda_t = 0.19245$ for both senses. (MODELLED, B)

a	Prograde			Retrograde		
	r_c	Ω_c	λ_t	r_c	Ω_c	λ_t
0.5	2.347	0.2441	0.1859	3.532	-0.1629	0.1901
0.9	1.558	0.3516	0.1361	3.910	-0.1464	0.1866
0.99	1.168	0.4441	0.0597	3.991	-0.1432	0.1857

Notes: Frequencies are signed by sense of rotation. Prograde λ_t/Ω_c falls from 1 at $a = 0$ to 0.134 at $a = 0.99$; retrograde λ_t moves by less than 3% over the full spin range. Source: `code/exp_bh.py`, run E3.

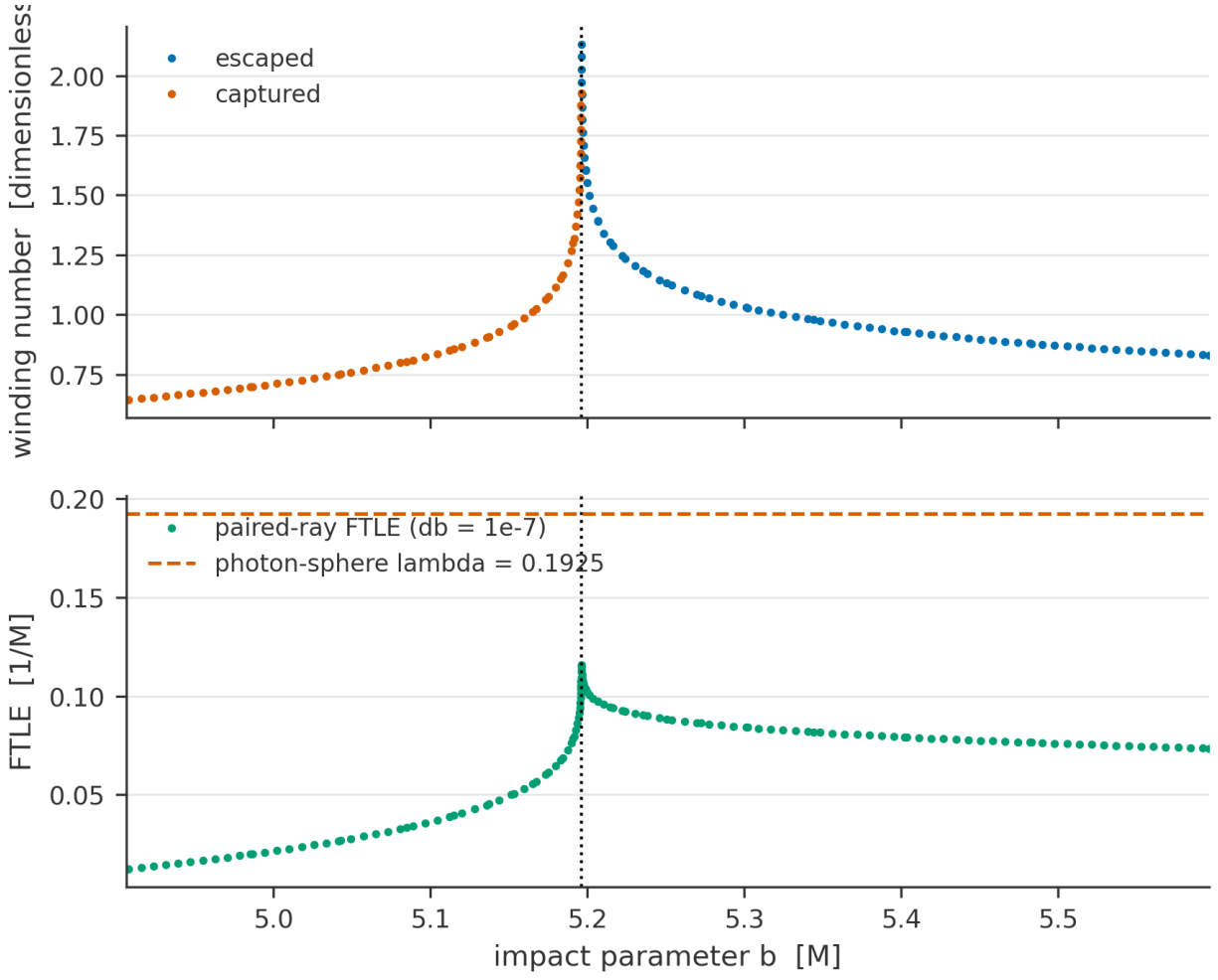


Figure 2. Winding number and finite-time Lyapunov exponent against impact parameter, Schwarzschild. Source: `code/exp_bh.py`, run E2. Note: stacked panels share the b axis; the FTLE is a whole-trajectory average (see text).

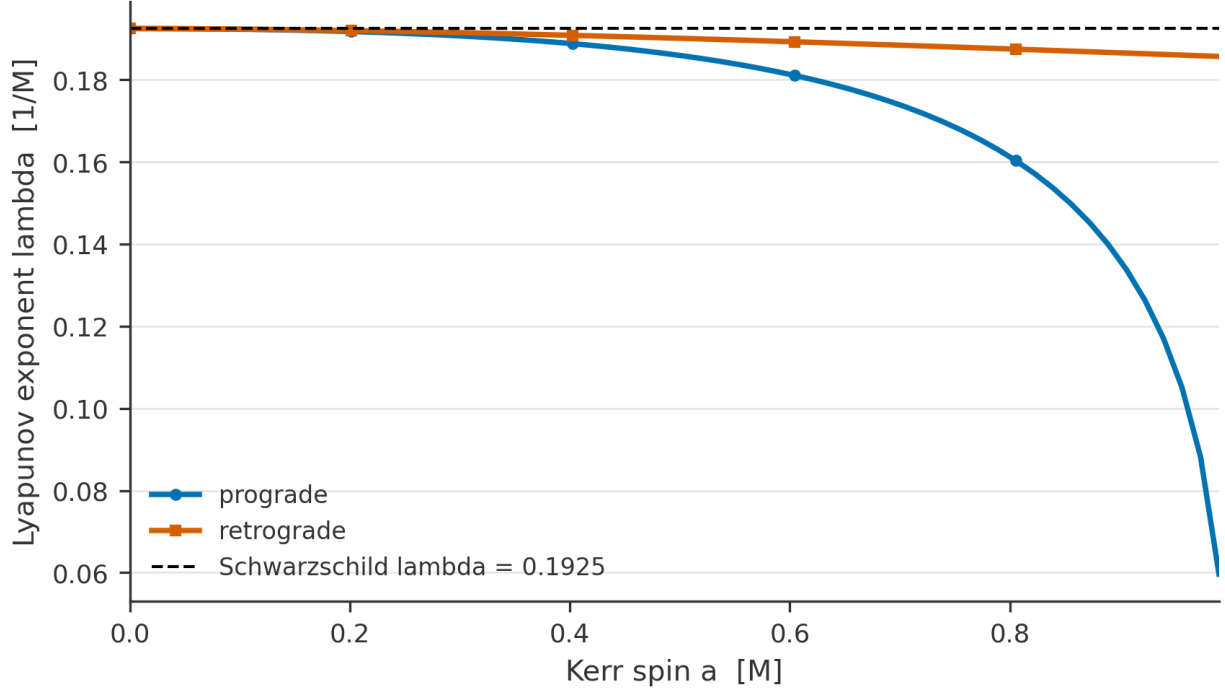


Figure 3. Lyapunov exponent against spin for Kerr equatorial photon circles. Source: `code/exp_bh.py`, run E3.

order of magnitude across the spin range (Figure 3). A caution from the imaging literature applies: the ring in current horizon-scale images is dominated by direct and lensing-ring emission rather than by the photon ring itself (Gralla et al., 2019; Cunha and Herdeiro, 2018), so the exponents here constrain strong-field structure and future photon-ring measurements rather than existing images.

3.3 Predictability horizons

With initial-condition uncertainty $\delta_0 = 10^{-8}$ (a stated, illustrative precision (ASSUMED, C)), (4) with the orbit exponent gives $T_{\text{pred}} = 59.8 M$ at tolerance $\Delta = 10^{-3}$ and $83.8 M$ at $\Delta = 10^{-1}$. In winding units the statement is cleaner: a near-critical Schwarzschild ray remains predictable for $n_{\text{max}} = \ln(\Delta/\delta_0)/(2\pi) \approx 1.8$ turns at the tighter tolerance, because $\lambda_t/\Omega_c = 1$. For the near-extremal prograde Kerr circle the same precision buys $n_{\text{max}} \approx 13.7$ turns ($\lambda_t/\Omega_c = 0.134$): weak instability, not precision, is what extends the horizon there. Using the whole-trajectory FTLE of run E2 in place of the orbit exponent lengthens the nominal horizon (median $122 M$ near b_c at $\Delta = 10^{-3}$), but that substitution is not correct for near-critical rays, whose dwell is governed by the orbit exponent; we report both numbers to make the distinction visible rather than to offer a choice between them.

3.4 Fisher information: what a point lens lets one recover

For a Schwarzschild point lens in the standard formalism (Schneider et al., 2006), with observables $\theta_{\pm}$ and relative time delay, Gaussian noise $\sigma_{\theta} = 10^{-3} \theta_E$ and $\sigma_t = 10^{-3} \tau$ (illustrative normalizations (ASSUMED, C)), the Cramér–Rao bound on the source angle β saturates as $\beta \rightarrow 0$: $\sigma_{\beta} \geq 4.7 \times 10^{-4}$

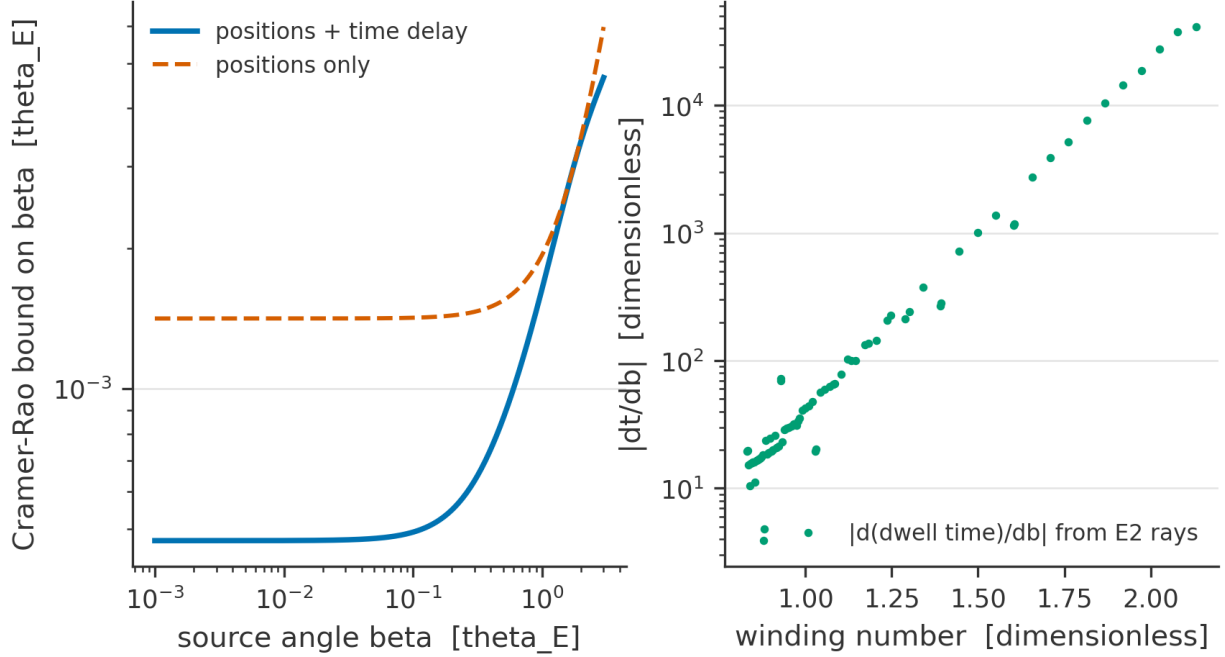


Figure 4. Cramér–Rao bound on the source angle against β (left), and arrival-time sensitivity $|dT/db|$ against winding number (right). Source: `code/exp_bh.py`, runs E2 and E4. Note: noise levels are the illustrative normalizations of the text; the right panel’s fitted slope is 6.323 against the analytic 2π .

at $\beta = 0.01$, essentially unchanged at $\beta = 0.1$, then degrading linearly, 1.6×10^{-3} at $\beta = 1$ and 4.7×10^{-3} at $\beta = 3$ (MODELLED, B). The time-delay observable dominates the positions at small β , improving the bound by a factor of about three there. Two lessons follow. First, in the integrable regime, approach to the caustic does not buy unbounded localization: the bound flattens because the image configuration approaches the Einstein ring, whose parameter derivatives remain finite. The exponential amplification of Section 2.6 is a property of near-critical *scattering* rays (successive windings), not of the two-image lens configuration. Second, timing is the sensitive channel, which is what the next subsection exploits.

3.5 Chaos-assisted sensitivity, measured on integrable rays

The quantity that grows without bound is the arrival-time response of a ray to its own impact parameter. Binning the run-E2 rays by winding number, the median $|dT/db|$ rises from 19.8 at $n \approx 0.9$ to 211 at $n \approx 1.25$ to 3.76×10^4 at $n \approx 2.1$, and $\ln |dT/db|$ is linear in n with fitted slope 6.323, against the analytic prediction $2\pi\lambda_t/\Omega_c = 2\pi = 6.283$ (+0.6%) (MODELLED, B). The derivation is one line from (5): $dT/db \sim 1/(\lambda_t|b - b_c|)$ and $|b - b_c| \sim e^{-2\pi\lambda_t n/\Omega_c}$. Each additional winding multiplies timing sensitivity by $e^{2\pi\lambda_t/\Omega_c}$, a factor of 535 for Schwarzschild. Translating dT/db into source-localization information requires identifying a source displacement with an impact parameter shift, $\beta_{\text{proxy}} \equiv b$; that identification is a modelling step, exact for a distant source translated across the lens axis in the scattering plane, and we flag it wherever it is used. The cost side of the ledger is the window: the set of impact parameters achieving n windings has width $\sim e^{-2\pi\lambda_t n/\Omega_c}$. Section 6 multiplies the two sides together.

4 Two-body scattering: the di-hole repeller

4.1 Setup and integrator validation

The Majumdar–Papapetrou di-hole is the simplest non-integrable relativistic lens: two extremal black holes of mass $M_1 = M_2 = 0.5$ held in static equilibrium at $z = \pm d/2$, with $U(\mathbf{x}) = 1 + M_1/|\mathbf{x} - \mathbf{x}_1| + M_2/|\mathbf{x} - \mathbf{x}_2|$ and metric $ds^2 = -U^{-2}dt^2 + U^2d\mathbf{x}^2$. (Shipley and Dolan, 2016) showed its photon dynamics carries a chaotic saddle with symbolic dynamics modelled on the three-disc problem, and conjectured Wada basin structure (Aguirre et al., 2001). What the literature does not contain (Section 6) is any of the standard quantitative diagnostics for that saddle: no uncertainty exponent, no basin entropy, no escape rate. This section computes the first two and reports a bounded failure on the third.

Rays are integrated in the coordinate-time (optical) parametrization $d\mathbf{x}/dt = \mathbf{p}/U^4$, $d\mathbf{p}/dt = \nabla U \cdot (U + |\mathbf{p}|^2/U^3)/U^2$, with a batched fixed-step RK4 in numpy ($h_t = 0.05$). The affine form of the same system was rejected after a step study: at fixed step it drifts by $O(1)$ in the constraint for plunging rays and misclassifies outcomes, while the optical form reproduces identical outcomes at every $h_t \in [0.005, 0.1]$ with constraint drift $\leq 3.4 \times 10^{-8}$. Capture is declared within 0.08 of a centre, escape at $|\mathbf{x}| > 80$, and the integration cap is $T_{\max} = 350$. Mirror symmetry provides an exact check: the $d = 2$ sweep returns 844 captures by each hole and 156 escapes each way, symmetric to the ray (MODELLED, B).

4.2 A fractal boundary on a thin set

The one-dimensional scattering function (2000 rays along $+x$ from $x = -40$, impact parameter $z \in (-4, 4)$) is mostly tame: outcomes change at only 27 points. The changes are not isolated, however; they concentrate in two thin bands, an inter-hole band at $|z| \approx 0.38$ – 0.45 and an outer rim band at $|z| \approx 3.70$ – 3.90 , and within the bands the transition count keeps rising under refinement: a window 0.04 wide still contains 17 transitions at spacing $dz = 2.7 \times 10^{-5}$ (Figure 5). This is the signature of a fractal boundary of small measure, not of global chaos.

The uncertainty exponent quantifies it. The grid-window census over $\epsilon \in [0.004, 0.128]$ fits $f(\epsilon) \sim \epsilon^\alpha$ with $\alpha = 0.669$ ($R^2 = 0.998$), hence boundary dimension $D = 1 - \alpha = 0.33$ on the one-dimensional cut, and $D = 2 - \alpha = 1.33$ for the two-dimensional basin by the codimension relation (MODELLED, B). Perturbation probes (500 base points, ϵ down to 10^{-5}) return $\alpha = 0.765$ but saturate at the $1/500$ counting floor for $\epsilon \leq 10^{-3}$; we take the census value as primary and the probe as a consistency bound. The control experiment is decisive: the same protocol on a single centre ($M = 1$) gives a smallest-scale local slope of exactly 1.000 (regular, $D = 0$) against the di-hole’s 0.710 at the same scales (Figure 6). The di-hole boundary is fractal; the single-hole boundary is not; the difference is the second centre, not the numerics. At separation $d = 3$ the structure persists with the inter-hole band moved to $|z| \approx 0.72$ – 0.82 and a weaker-fit $\alpha = 0.585$ ($R^2 = 0.962$, flagged as less secure).

4.3 Basins, dwell times, and a null on the escape rate

The two-dimensional exit basin (rays from $x = -40$, impact parameters (y, z) on a 160×160 grid) shows two capture disks and cross-capture crescents at the far rims: rays aimed near one hole can be captured by the other (Figure 7). The disks are centred well outside the hole positions marked

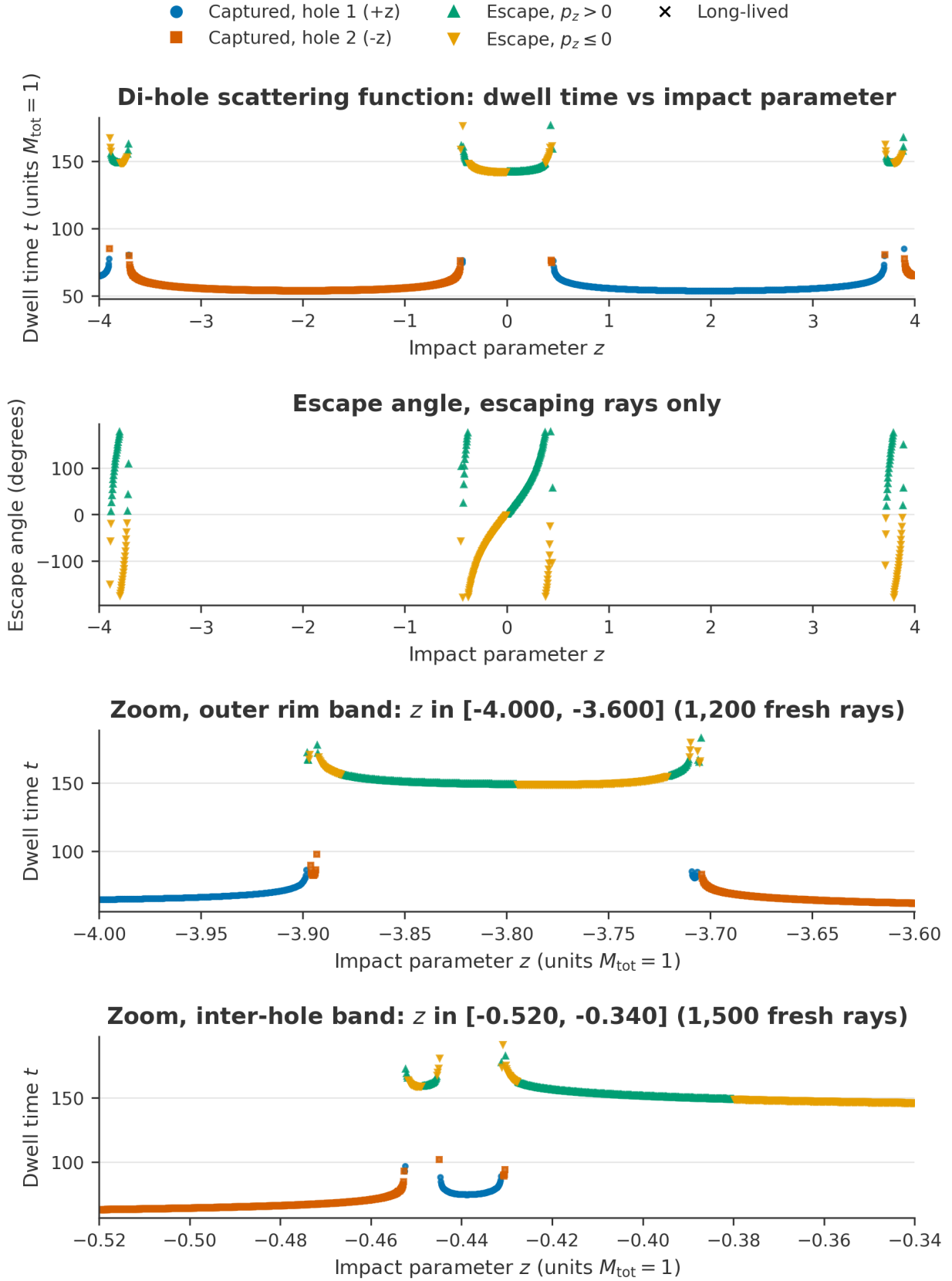


Figure 5. Di-hole scattering function at $d = 2$ with a zoomed window. Source: `code/exp_dihole.py`, run D1.

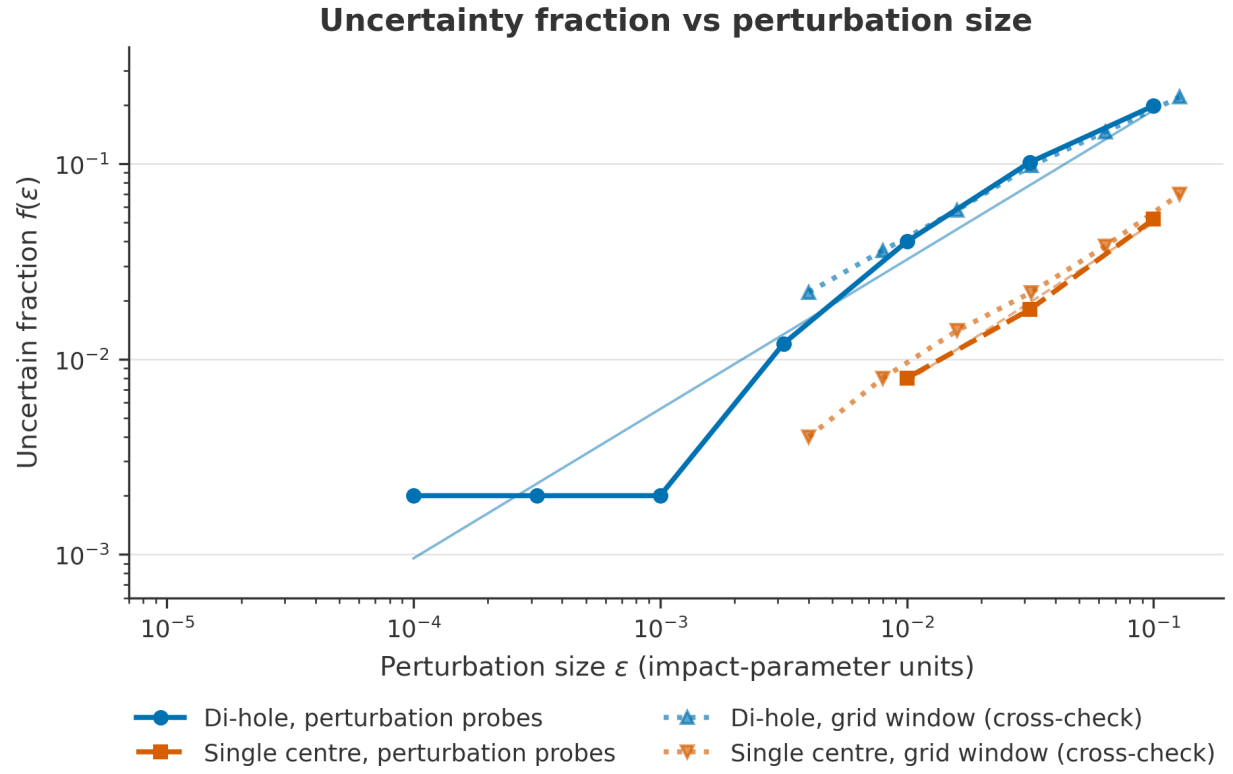


Figure 6. Uncertainty fraction against perturbation size, di-hole against single-centre control. Source: `code/exp_dihole.py`, runs D2 and D4. Note: log-log scales; fitted slopes are the uncertainty exponents.

in the figure: a ray aimed directly at one hole is deflected by the companion, and capture requires aiming wide of the target, a strong-field effect with no single-hole analogue. Only 1.7% of adjacent grid pairs disagree in outcome, again the picture of a thin uncertain set. Dwell times are bimodal: captures dwell ≈ 55 time units, escapes ≈ 140 – 150 , and 44.2% of escapes are backscattered. The maximum dwell in any run is 183.4.

The null result is this: in roughly 51,000 rays across all samples, *none* is long-lived against the cap $T_{\max} = 350$. There is therefore no exponential dwell-time tail to fit, and the escape rate γ of Section 2.5 cannot be measured for this system at these parameters; the escape-rate relation test cannot be executed. We report the failure rather than a number. Its meaning is physical, not merely numerical: along this incidence geometry the di-hole saddle traps rays only transiently. The fractal boundary is the shadow of a saddle that the rays of this family barely visit, which is consistent with the thinness of the uncertain set ($\alpha = 0.669$ corresponds to an uncertain fraction of only $\approx 1\%$ at resolution $\epsilon = 10^{-3}$). A geometry that aims rays along the inter-hole axis, where (Shipley and Dolan, 2016) find the strongly chaotic region, is the obvious next experiment and is outside this paper’s computed set.

4.4 What survives coarse-graining

Chaotic unpredictability is local in initial-condition space, and its complement is measurable. Predicting each fine-grid outcome by the majority outcome of its coarse cell gives accuracy 0.814 at cell width 1.0, 0.970 at 0.1, and 0.993 at 0.01, against a no-information baseline of 0.422; the mean within-cell outcome entropy falls from 0.659 bits at width 1.0 to 0.017 bits at width 0.01 (MODELLED, B) (Figure 8). Two readings matter for Section 6. First, “unpredictable” is a statement about a resolution: at $\epsilon = 10^{-2}$ the di-hole scattering function is 99.3% predictable by coarse-graining, and the irreducible residue is exactly the fractal set of dimension 0.33. Second, the outcome statistics retain structure that individual outcomes lose: the majority map at width 0.1 already recovers the capture disks, the crescents, and the band locations. Statistical recoverability and pointwise predictability are different goods, and chaos taxes only the second.

5 Wormholes, higher dimensions, and media

5.1 Ellis wormhole: an integrable throat with a black hole’s ratio

The Ellis wormhole (Ellis, 1973), the static member of the traversable family of (Morris and Thorne, 1988), is spherically symmetric and hence integrable: like Schwarzschild it is unstable, not chaotic. Its throat is an unstable photon orbit with $b_c = a$ and, as derived in Appendix A, $\lambda_t = \Omega_c = 1/a$. Deviation integration at $a = 1$ returns $\lambda_t = 0.99986$, a relative error of 1.4×10^{-4} (MODELLED, B). The scattering phenomenology matches: winding number and whole-trajectory FTLE both peak sharply at b_c (maximum winding 1.79 turns at $b = 1.0001$; Figure 9). Measured against the invariant of Section 2.4, the throat is indistinguishable from a Schwarzschild photon sphere: $\lambda_t/\Omega_c = 1$ in both. What differs is the outcome set. A near-critical Schwarzschild ray is captured or escapes; a near-critical Ellis ray is transmitted to the other asymptotic region or returns, and the throat orbit separates those outcomes. Unlike the paired light rings of generic horizonless ultracompact objects (Cunha et al., 2017), the Ellis potential has a single extremum, so the throat orbit is the only one. No chaotic scattering treatment of any wormhole exists in the literature (Section 6);

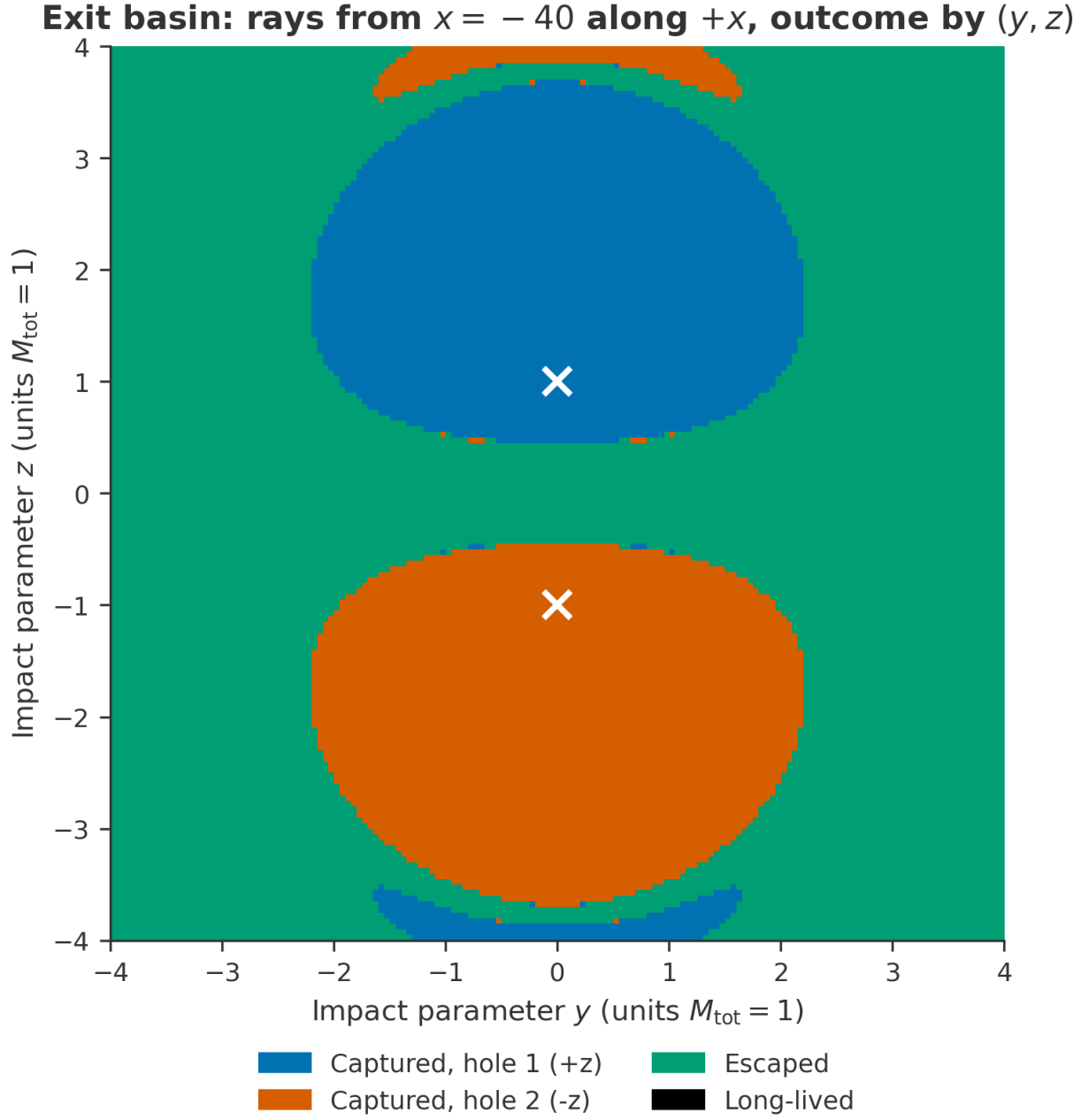


Figure 7. Exit basin of the di-hole at $d = 2$ in the impact plane. Source: `code/exp_dihole.py`, run D3.

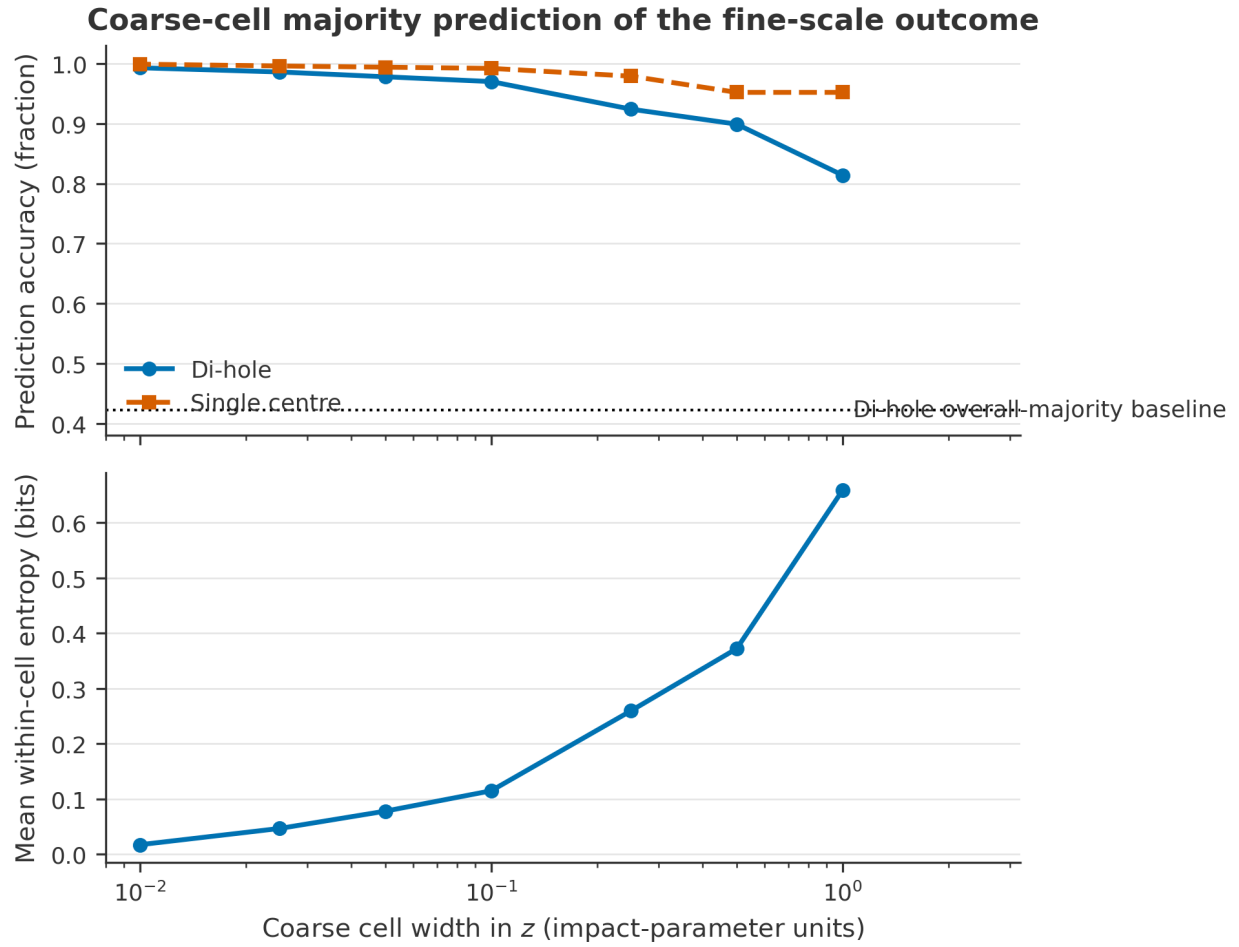


Figure 8. Coarse-graining recovery of di-hole outcomes: majority-prediction accuracy and within-cell entropy against cell width. Source: `code/exp_dihole.py`, run D5.

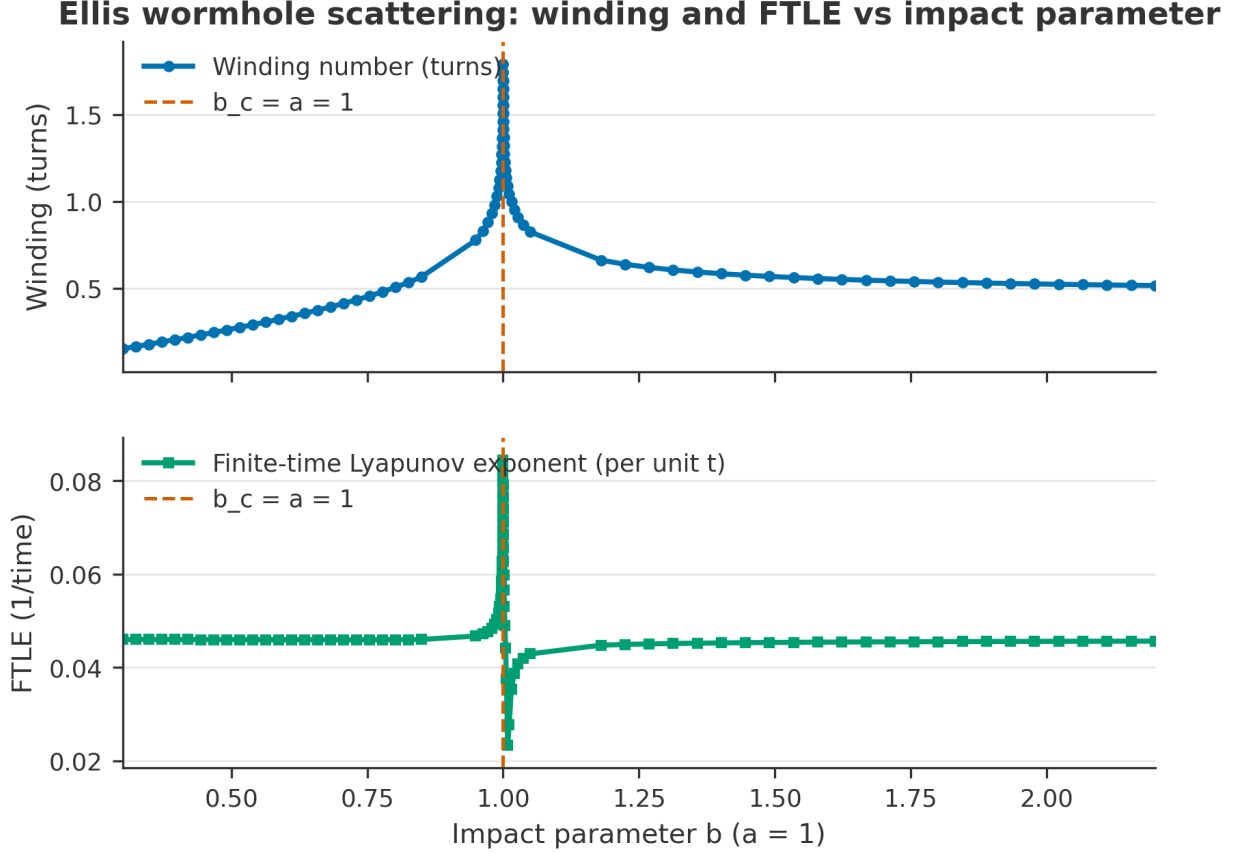


Figure 9. Winding number and finite-time Lyapunov exponent against impact parameter for the Ellis wormhole, $a = 1$. Source: `code/exp_media.py`, run W1. Note: the FTLE is a whole-trajectory average diluted by free flight; the peak at b_c is the comparable quantity.

wormhole lensing remains weak-field (Abe, 2010; Cramer et al., 1995). The Ellis case shows why the single-throat geometry cannot supply one, and points to multi-orbit wormholes, including rotating ones (Teo, 1998), as the open case.

5.2 Higher dimensions: instability per turn grows as $\sqrt{d-3}$

Proposition A.1 gives $\lambda_t/\Omega_c = \sqrt{d-3}$ for Tangherlini black holes (Tangherlini, 1963). Table 2 evaluates the closed forms at $r_s = 1$; deviation integration at $d = 5$ returns $\lambda_t = 0.7063$ against the analytic 0.7071 (0.11%) (MODELLED, B). The content for this paper is the ratio's growth: in higher dimensions each turn around the photon sphere amplifies deviations by $e^{2\pi\sqrt{d-3}}$ rather than $e^{2\pi}$, while the winding coefficient $A = \Omega_c/(2\pi\lambda_t) = 1/(2\pi\sqrt{d-3})$ shrinks: fewer turns per decade of approach, more sensitivity per turn. Dimension is thus a clean control on the tradeoff of Section 6, in the integrable regime. Chaotic scattering in $d > 4$ has not been computed anywhere (Section 6); the integrable baseline is what this paper adds.

Table 2. Tangherlini photon spheres at $r_s = 1$: radius r_{ph} , critical impact parameter b_c , orbital frequency Ω_c , Lyapunov exponent λ_t . (RECONSTRUCTED, A)

d	r_{ph}	b_c	Ω_c	λ_t	λ_t/Ω_c
4	1.500	2.598	0.385	0.385	1.000
5	1.414	2.000	0.500	0.707	1.414
6	1.357	1.752	0.571	0.989	1.732
7	1.316	1.612	0.620	1.241	2.000

Notes: $\Omega_c = 1/b_c$ identically here; $\lambda_t = \sqrt{d-3} \Omega_c$ by Proposition A.1. The $d = 4$ row is Schwarzschild at $M = 1/2$. Source: `code/exp_media.py`, run W2.

Higher-dimensional Schwarzschild: Lyapunov exponent vs dimension

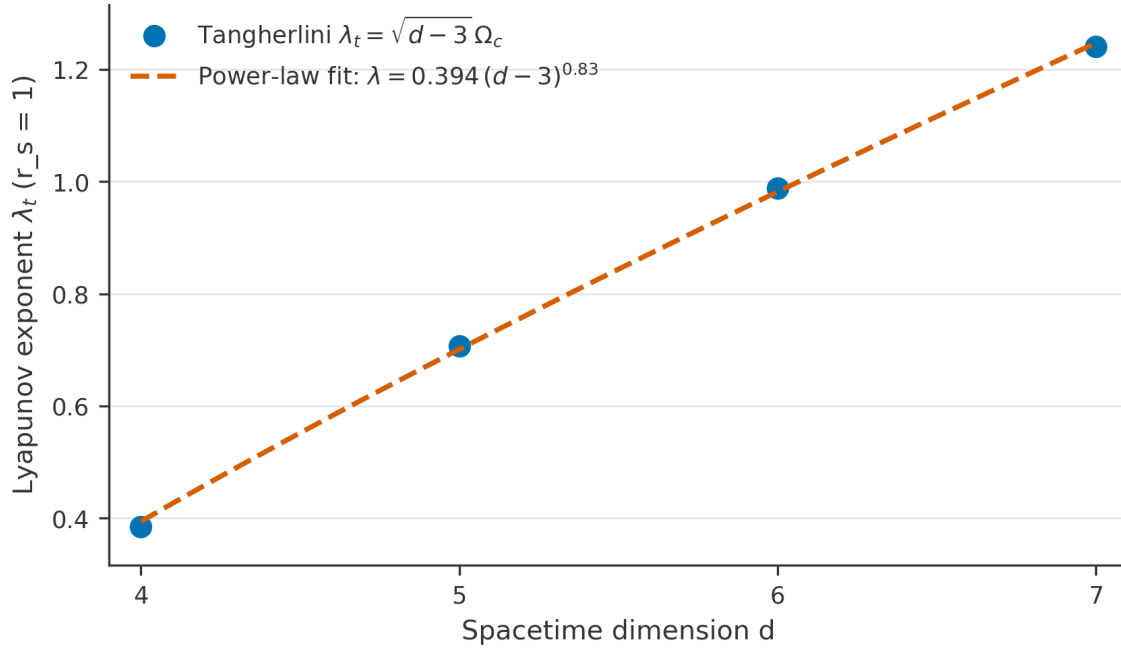


Figure 10. Lyapunov exponent against spacetime dimension, Tangherlini, with the $\sqrt{d-3}$ law of Proposition A.1. Source: `code/exp_media.py`, run W2.

5.3 Plasma lenses: the regular side of the transition, and a null

In a plasma the rays are not spacetime null geodesics but Hamiltonian rays of the refractive index field, and deflection is chromatic (Bisnovatyi-Kogan and Tsupko, 2009, 2010; Perlick, 2000). Our medium is flat spacetime with $n^2(\mathbf{x}) = 1 - A \sum_i \exp(-|\mathbf{x} - \mathbf{x}_i|^2/2\sigma^2)$: underdense Gaussian lenses that repel rays. A single lens ($A = 0.5$, $\sigma = 1$) gives a smooth scattering function with maximum exit angle 0.439 rad: regular, as a single smooth deflector should be. The double lens at the same parameters is more structured (four extrema against two) but remains smooth, and its uncertainty exponent is $\alpha = 1.01$: the regular value (MODELLED, B). At these parameters there is no trapped set (the global minimum of n^2 sampled by the rays is 0.494, and no ray is reflected or long-lived), so the system sits on the regular side of the transition that the di-hole crosses. We report this as a bounded null, not as evidence that plasma ray chaos does not exist. A follow-up scan (same integrator and protocol; `code/exp_plasma_scan.py`) extends the coupling to $A \in \{0.7, 0.85, 0.95\}$ at separation 3σ and $A \in \{0.85, 0.95\}$ at 2σ , and finds no regular-to-chaotic transition anywhere in the scanned range. At 2σ a forbidden region ($n^2 \leq 0$) switches on between the lenses: no ray enters it (rays turn at its boundary), 276–396 of 2000 rays backscatter, and the unwrapped exit angle sweeps smoothly and monotonically through ≈ 5.97 rad, a full-circle deflection. Long dwell times at 2σ are steep drifters crossing the sampling plane, verified by re-integration not to be trapped orbits. The untested region is now specific: separations slightly above 2σ , where the forbidden region splits into two islands between which a ray could in principle bounce, and $A > 0.95$.

The scan also exposes a trap in the diagnostic itself. The naive uncertainty-fraction protocol returns $\alpha = 0.76$ – 0.83 in four of the five strong-coupling sets, which a screening criterion would read as chaos, including at $A = 0.7$, 3σ , where the scattering function is demonstrably smooth. Two controls assign the deficit to the estimator: bin saturation at large ϵ in the binned exit-angle outcome. A smooth-slope control reproduces most of the deficit from the measured $\theta'(b)$ alone, and high-statistics asymptotic fits at $\epsilon = 10^{-3}$ – 10^{-4} return $\alpha = 0.95$ – 1.09 for all five sets (Figure 11). All verdicts in this section use the adjudicated values. The lesson transfers: uncertainty exponents estimated from binned continuous outcomes at coarse ϵ are biased toward chaos, and a fractal claim needs the smallest-scale behaviour, a smooth control, or discrete outcomes. The di-hole measurement of Section 4 uses discrete outcome classes with a same-protocol control and is not subject to this bias.

One further result cuts against a natural hope. The single lens has an angular focusing configuration where $d\theta/db$ changes sign ($b_{\text{focus}} = 0.70$), the ray-optics analogue of the synchronization sought in (Kilambi Research, 2026). But the optical path length spread across a bundle of width 0.1 centred there is 0.0151, against 0.00028 for a generic bundle at $b = 3.5$: arrival times are fifty-four times *less* synchronized at the angular focus (MODELLED, B). Angular stationarity and temporal stationarity are different conditions in a dispersive medium, and the first does not buy the second.

6 Synthesis: chaos, focusing, and information

6.1 The comparison

Table 3 assembles the computed set on standardized rows. Read across the λ_t/Ω_c column first: it is 1 for Schwarzschild, for the Ellis throat, and (by near-cancellation) for retrograde Kerr; it collapses to 0.134 for near-extremal prograde Kerr; it grows as $\sqrt{d-3}$ in higher dimensions. This one ratio sets the winding coefficient $A = \Omega_c/(2\pi\lambda_t)$, the sensitivity amplification per turn $e^{2\pi\lambda_t/\Omega_c}$, and the number of predictable turns at fixed precision, $n_{\text{max}} = \ln(\Delta/\delta_0)/(2\pi\lambda_t/\Omega_c)$. Read the α column

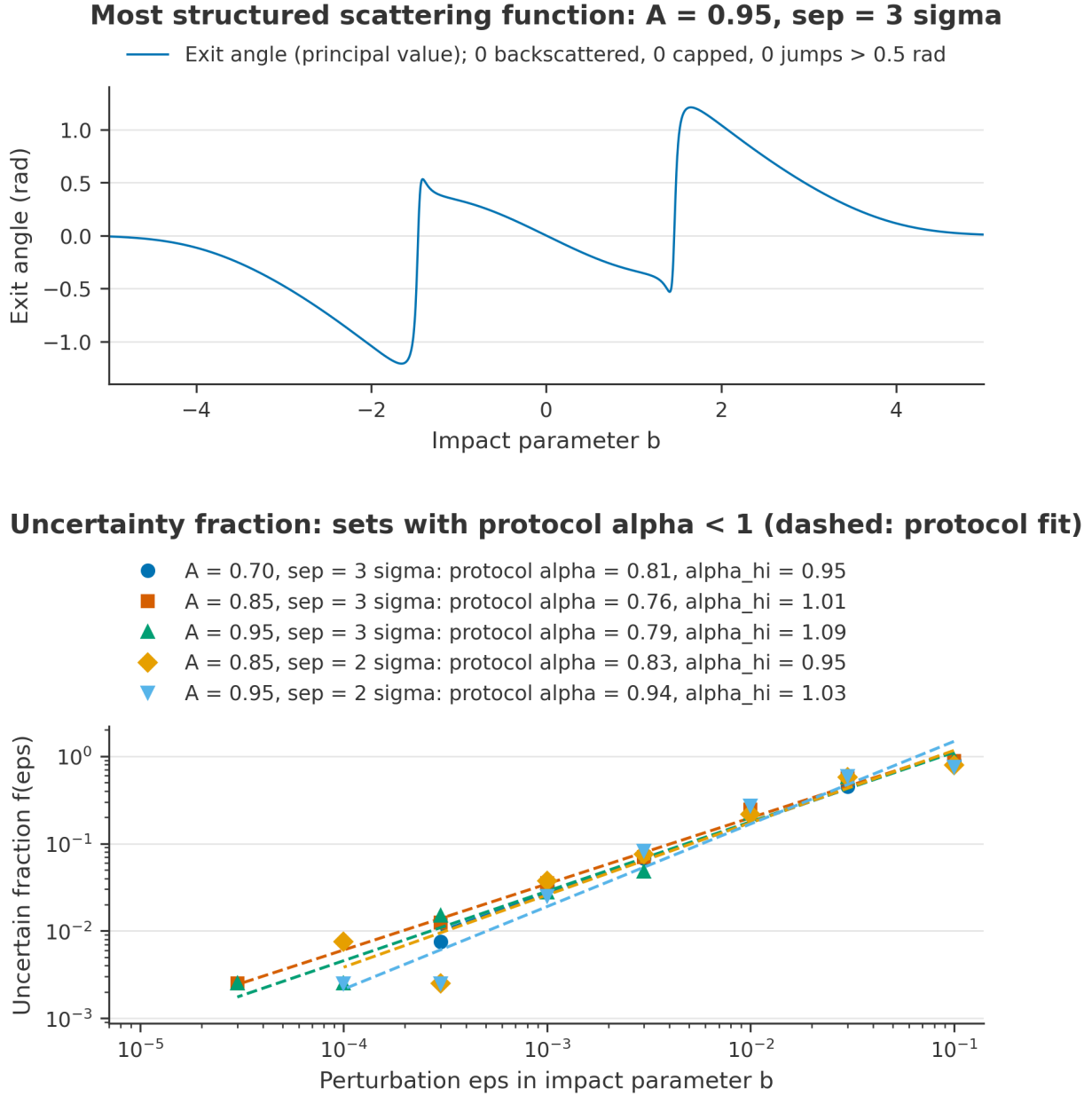


Figure 11. Plasma scan: scattering function for the most structured set ($A = 0.95$, separation 3σ), and protocol uncertainty exponents against ϵ for all five sets. Source: `code/exp_plasma_scan.py`. Note: the protocol exponents at coarse ϵ are estimator-biased (see text); adjudicated asymptotic values are 0.95–1.09.

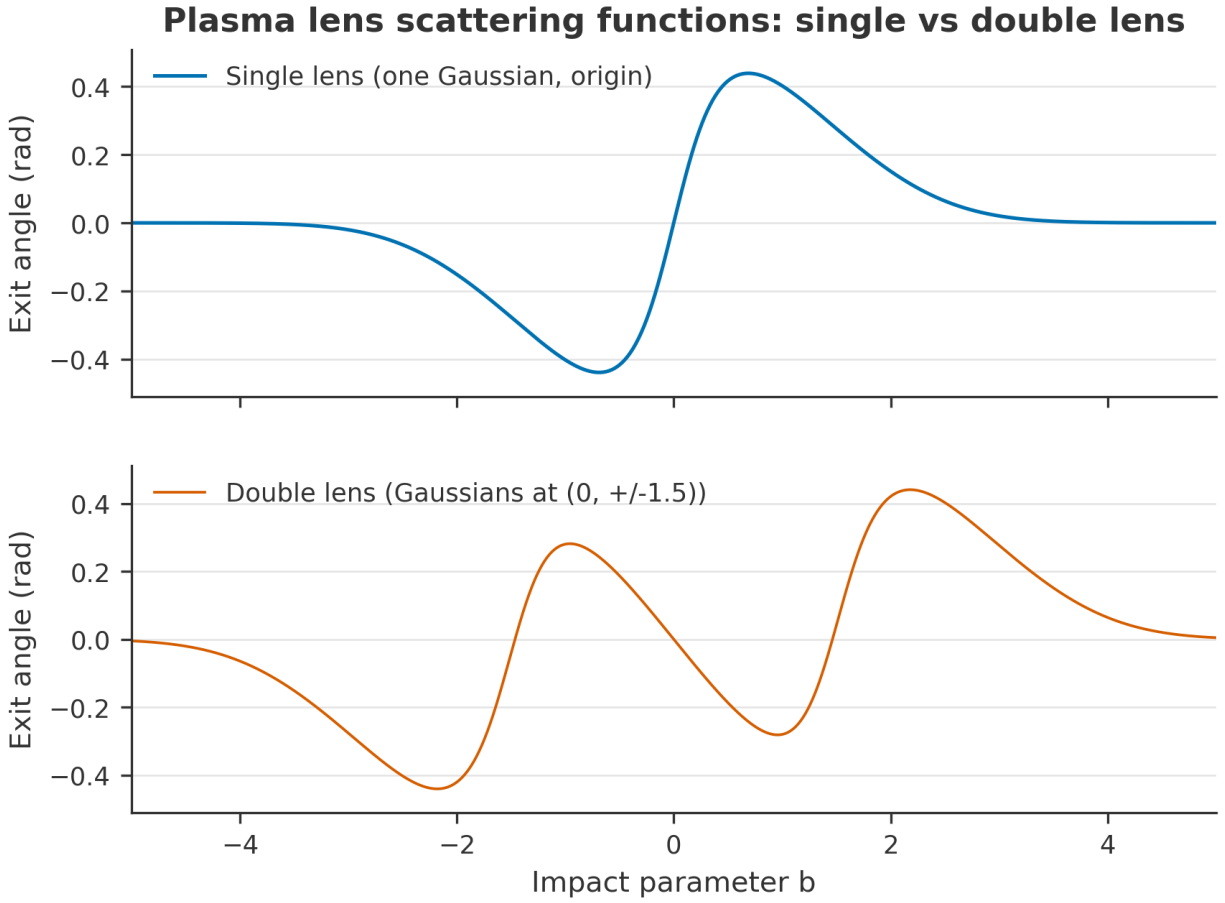


Figure 12. Exit angle against impact parameter for single and double plasma lenses, $A = 0.5$. Source: `code/exp_media.py`, run P1.

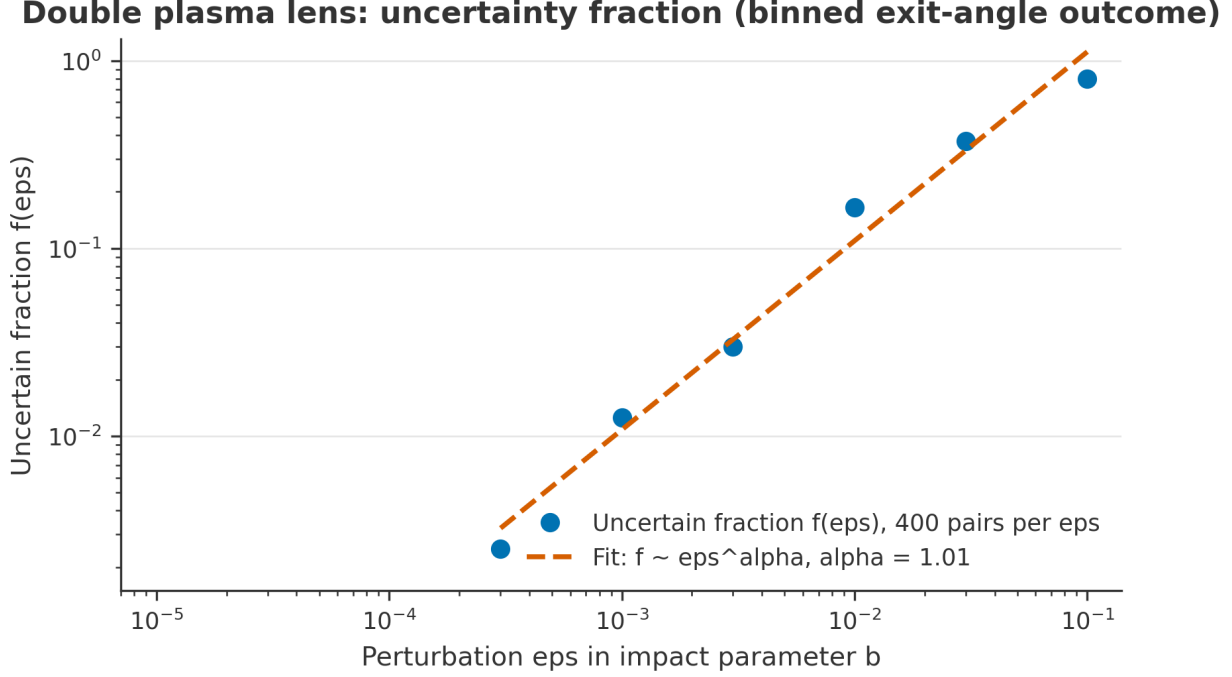


Figure 13. Uncertainty fraction against perturbation size for the double plasma lens, with the fitted exponent. Source: `code/exp_media.py`, run P1.

second: it separates the regimes. Every integrable system has, in effect, α undefined or 1 (a single critical point, a smooth control); the di-hole has $\alpha = 0.669$; the double plasma lens at $A = 0.5$ has $\alpha = 1.01$ and sits with the regular systems despite its second deflector.

6.2 The chaos–information tradeoff

The measurements support a precise statement of when sensitivity helps reconstruction. Group near-critical rays into winding shells: shell n collects rays with winding number in $[n, n + 1)$. From (5), the shell’s width in impact parameter and the arrival-time sensitivity within it scale as

$$w_n \propto e^{-2\pi(\lambda_t/\Omega_c)n}, \quad S_n \equiv \left| \frac{dT}{db} \right|_n \propto \frac{1}{\lambda_t} e^{+2\pi(\lambda_t/\Omega_c)n}, \quad (6)$$

of which the second is measured on Schwarzschild rays with fitted exponent 6.323 against 2π predicted (Section 3.5). Two products follow. The sensitivity–window product $w_n S_n \propto 1/\lambda_t$ is invariant: geometric optics conserves the total timing displacement that a unit interval of impact parameter can be made to express. The Fisher product behaves differently: with timing noise σ_t independent of n , the expected Fisher information per unit impact-parameter interval in shell n is $\mathcal{I}_n \propto w_n S_n^2 / \sigma_t^2 \propto \lambda_t^{-2} e^{+2\pi(\lambda_t/\Omega_c)n} / \sigma_t^2$, which grows with n .

Conjecture 6.1 (Chaos–information tradeoff). In geometric optics, expected arrival-time Fisher information per unit source-plane area from near-critical scattering grows as $e^{+2\pi(\lambda_t/\Omega_c)n}$ with winding number n , and is cut off at the first of three scales: wave-optics smoothing of the scattering function, finite source size (which smears shells narrower than the source), and the observation window (dwell time $\propto n$). Chaotic or unstable sensitivity is a net resource for source localization

Table 3. Standardized comparison of computed systems. Exponents in coordinate time t ; $M = 1$ except Tangherlini ($r_s = 1$) and Ellis ($a = 1$). (MODELLED, B) except where noted.

System	Regime	λ_t	λ_t/Ω_c	$A = \Omega_c/2\pi\lambda_t$	Chaos diagnostic
Schwarzschild	unstable	0.1925	1.000	0.159	single critical b_c
Kerr prograde, $a = 0.99$	unstable	0.0597	0.134	1.184	single critical b_c
Kerr retrograde, $a = 0.99$	unstable	0.1857	1.297	0.123	single critical b_c
Ellis wormhole	unstable	1.000	1.000	0.159	single critical b_c
Tangherlini $d = 5$	unstable	0.707	1.414	0.113	single critical b_c
Tangherlini $d = 7$	unstable	1.241	2.000	0.080	single critical b_c
Di-hole, $d = 2$	chaotic (thin)	n/a	n/a	n/a	$\alpha = 0.669$, $D_{1d} = 0.33$
Double plasma (scan)	regular	n/a	n/a	n/a	$\alpha = 0.95$ – 1.09 (asymptotic)

Notes: Schwarzschild λ_t is (RECONSTRUCTED, A) (closed form) and verified numerically to 0.074%; Ellis λ_t is the closed form of Appendix A, verified to 0.014%. Tangherlini rows by Proposition A.1. Sources: `code/exp_bh.py`, `code/exp_dihole.py`, `code/exp_media.py`.

exactly when the observable is an arrival time or phase, the source is effectively pointlike against the shell width, and the cutoff shell lies above the shell carrying the signal.

The integrable lens of Section 3.4 is the control case: with no winding shells, the Cramér–Rao bound saturates at the caustic rather than improving, so the growth in the conjecture is attributable to the shells, not to caustic approach in general. For the chaotic regime the analogous statement is statistical rather than pointwise: outcome information at resolution ϵ is lost only on the uncertain fraction ϵ^α (Section 4.4), so a chaotic saddle taxes reconstruction by a factor governed by α , while its basin statistics remain recoverable at any fixed $\epsilon > 0$. Pointwise prediction and statistical recovery are taxed by different exponents (λ_t and α respectively), and a geometry can be expensive in one currency and cheap in the other: the di-hole is exactly that case.

For inverse design the tradeoff supplies an objective rather than a slogan. To maximize localization information per source at fixed observing time: choose the observable channel with derivatives in the winding shells (arrival times, phases), maximize λ_t/Ω_c subject to keeping the working shell below the wave and source-size cutoffs, and keep α near 1 if outcome reliability matters more than sensitivity. To maximize synchronization, the plasma result of Section 5.3 is the warning: angular focusing conditions and arrival-time stationarity are independent requirements, and a design that optimizes the first can degrade the second by orders of magnitude. A full inverse-design computation against this objective is future work; the companion paper’s index-profile optimization (Kilambi Research, 2026) is the integrable-regime instance.

6.3 Established, synthesis, new

Established (cited, not claimed). The photon-orbit exponent formula and the eikonal quasinormal-mode correspondence (Ferrari and Mashhoon, 1984; Cardoso et al., 2009); Kerr integrability (Carter, 1968); the logarithmic strong-deflection divergence (Virbhadra and Ellis, 2000; Bozza, 2002); photon-ring critical parameters as the instability exponent in observational form (Gralla and Lupsasca, 2020a; Johnson et al., 2020); the di-hole chaotic saddle and its symbolic dynamics (Shipley and Dolan, 2016); the uncertainty exponent and basin entropy formalisms (Grebogi et al., 1987; Daza et al., 2016); the escape-rate formalism (Ott and Tél, 1993); chromatic plasma lensing (Bisnovatyi-Kogan

and Tsupko, 2009; Tsupko and Bisnovatyi-Kogan, 2017); the Ellis throat orbit (Ellis, 1973; Abe, 2010).

Synthesis (ours, assembled from established pieces). The four-regime classification applied uniformly across geometries and media; the use of λ_t/Ω_c as the organizing invariant linking winding counts, sensitivity amplification, and predictability horizons; the standardized comparison of Table 3; the reading of the di-hole coarse-graining measurement as statistical (as opposed to pointwise) information recovery.

New (candidate results, labelled by strength). (i) **COMPUTATION:** the first uncertainty exponent ($\alpha = 0.669$) and coarse-graining information measures for Majumdar–Papapetrou di-hole exit basins, with a single-centre control (local slope 1.000) isolating the second centre as the cause; the literature contains the saddle’s existence proof and shadow pictures but no such diagnostics (searches of 12 October 2026). (ii) **COMPUTATION:** the Ellis throat exponent in closed form, $\lambda_t = 1/a$ with $\lambda_t/\Omega_c = 1$, verified numerically; an elementary consequence of standard methods, claimed as a computation, not a theorem of note. (iii) **PROPOSITION** (proved here): $\lambda_t/\Omega_c = \sqrt{d-3}$ for Tangherlini photon spheres. The proof is a short calculation from (3); we have not found the ratio stated in this form and flag it as an elementary closed form rather than a deep result. (iv) **COMPUTATION:** the arrival-time sensitivity law $\ln|dT/db|$ linear in winding number with slope $2\pi\lambda_t/\Omega_c$, measured at 6.323 against 2π for Schwarzschild; the derivation from (5) is one line, so the content is the measurement and its Kerr and higher-dimensional scaling, not the algebra. (v) **CONJECTURE:** the chaos–information tradeoff above, with its three cutoffs. Open. (vi) **COMPUTATION** (nulls): the double plasma lens is regular at $A = 0.5$ ($\alpha = 1.01$) and across a five-set strong-coupling scan (adjudicated $\alpha = 0.95$ – 1.09), with backscattering but no trapping at 2σ separation; the scan additionally shows that the naive uncertainty-fraction estimator is biased toward chaos for binned continuous outcomes, a methods result with consequences beyond this paper. The single-lens angular focus desynchronizes arrival times by a factor of 54 against a generic bundle; the di-hole escape rate is not measurable at the computed parameters because no long-lived rays occur.

What was attempted and not achieved. A Fisher theory for the chaotic observables themselves (α , basin entropy, fractal dimension as estimators) does not exist in the literature and is not produced here: Fisher methods in this area exist only for smooth parametric image models (Farah et al., 2026). Ray-by-ray likelihoods are non-smooth on the fractal set, so the Cramér–Rao regularity conditions fail; the well-posed replacement (information in basin statistics over finite beams, or in ensemble observables of the kind (Hadar et al., 2021) construct for photon rings) is identified but not solved. The Melnikov-type criterion for chaos onset under Kerr perturbations likewise remains open.

7 Discussion

7.1 Falsifiers

The main claims fail in specific, checkable ways. (1) The winding law $A = \Omega_c/(2\pi\lambda_t)$ is fitted at $R^2 = 0.999997$ on both sides of b_c ; an independent integration at higher precision that moved the fitted coefficient outside 1% of $1/(2\pi)$ for Schwarzschild would break the framework’s central identity, not just a number in it. (2) The di-hole uncertainty exponent is falsifiable by incidence geometry: rays aimed along the inter-hole axis should, on the reading given here, produce a thicker uncertain

set and the long-lived rays this paper did not find; if they instead return $\alpha \approx 1$, the fractal boundary reported here is an artifact of the chosen family and the chaos claim for this configuration collapses. (3) Proposition A.1 is a proof within the stated metric family; its numerical instance at $d = 5$ could have failed and did not (0.11%). (4) The tradeoff conjecture is falsified by any geometric-optics system in which per-shell expected Fisher information fails to grow as $e^{+2\pi(\lambda_t/\Omega_c)n}$ before the stated cutoffs, or in which the cutoffs can be shown not to bind. (5) The plasma synchronization null is parameter-scoped: a plasma lens configuration exhibiting arrival-time stationarity at an angular focus would not contradict the measurement at $A = 0.5$ but would bound its generality, and is worth looking for.

7.2 Limitations

Geometric optics throughout: wave-optics smoothing of fractal scattering functions, which the tradeoff conjecture names as a cutoff, is asserted from the structure of the theory and not computed. The Fisher analysis uses illustrative noise normalizations ($\sigma_\theta = \sigma_t = 10^{-3}$ in lens units), so its bounds are scaling statements, not forecasts for an instrument. The translation from impact parameter to source displacement ($\beta_{\text{proxy}} \equiv b$) is exact only for a distant source translated across the axis in the scattering plane. The Kerr analysis is equatorial; spherical photon orbits off the equator have their own exponent spectrum, not computed here. The di-hole results are for one incidence family (rays along $+x$, holes on the z axis) at two separations, and the zero-long-lived-ray null is a property of that family until the inter-hole incidence is computed. The plasma work is a flat-spacetime analogue: frequency dependence (the chromatic deformation of a saddle, an open problem in the literature) is not scanned, and no claim transfers automatically to plasma in curved spacetime. Whole-trajectory FTLE values are diluted by free flight and are used only as locators of critical structure. Uncertainty exponents carry an estimator caveat established by the plasma scan: binned continuous outcomes bias α downward at coarse ϵ ; the di-hole values use discrete outcomes with a same-protocol control, and the plasma verdicts use adjudicated asymptotic fits, but replications should report the estimator protocol alongside the exponent. Finally, coordinate-time Lyapunov exponents are diagnostics tied to the asymptotic observer’s time (Section 2.2); the chaos claims rest on the parametrization-independent diagnostics, as they should.

8 Conclusion

Geometry governs light three times over: through the trajectories themselves, through their sensitivity to initial conditions, and through the information about the source that observations retain. This paper measured the second and third across black holes, a wormhole, higher dimensions, a two-body relativistic lens, and plasma media, with the regimes kept distinct. Instability is common and cheap: every photon sphere supplies it, with an invariant ratio λ_t/Ω_c that ranges from 0.134 (near-extremal prograde Kerr) through 1 (Schwarzschild, Ellis) to $\sqrt{d-3}$ in higher dimensions, and that fixes winding counts, predictability horizons in turns, and the per-turn amplification of arrival-time sensitivity (measured slope 6.323 against 2π). Chaos is rarer and more specific: in the di-hole it lives on a thin fractal set ($\alpha = 0.669$, $D_{1d} = 0.33$), taxes pointwise prediction only there, and leaves basin statistics recoverable at the 99% level at resolution 10^{-2} . Sensitivity is not the enemy of reconstruction: in the arrival-time channel, expected Fisher information per unit source area grows into the winding shells as $e^{+2\pi(\lambda_t/\Omega_c)n}$, cut off by wave optics, source size, and dwell time, while in position-only channels through a simple lens it saturates. The three open problems that would most

change this picture are a Fisher theory for chaotic observables, whose regularity obstruction is now stated precisely; an escape-rate measurement for the di-hole along its strongly chaotic incidence; and a Melnikov-type criterion that predicts chaos onset under Kerr perturbations instead of discovering it system by system.

Production note

This paper was produced by Kilambi Research’s quantitative research pipeline with AI assistance; all derivations were checked against the cited sources and all numerical results were computed from the stated equations with the documented methods and tolerances. No new observations are reported. This is a retrospective analysis; there was no pre-registered plan.

A Derivations

A.1 The winding invariant

For a ray with impact parameter b scattering off an unstable circular photon orbit, write the initial radial offset from the orbit’s stable manifold as $\delta_0 = C|b - b_c|$ with $C = O(1)$. The deviation grows as $\delta(t) \simeq \delta_0 e^{\lambda_t t}$ while the ray remains in the linear regime, and the ray leaves the neighbourhood of the orbit when $\delta = O(1)$, giving a dwell time $T \simeq \lambda_t^{-1} \ln(1/(C|b - b_c|))$. The azimuthal advance during the dwell is $\Delta\phi = \Omega_c T$, hence $n(b) = \Delta\phi/(2\pi) \simeq -(\Omega_c/(2\pi\lambda_t)) \ln|b - b_c| + O(1)$, which is (5). For Schwarzschild, $\Omega_c/\lambda_t = 1$ and the predicted fit coefficient is $A = 1/(2\pi) \approx 0.1592$.

A.2 Ellis wormhole throat exponent

The Ellis metric in proper radial coordinate l is $ds^2 = -dt^2 + dl^2 + (l^2 + a^2)d\Omega^2$. Null geodesics conserve $E = \dot{t}$ and $L = (l^2 + a^2)\dot{\phi}$, and the radial equation is $\dot{l}^2 = E^2 - L^2/(l^2 + a^2)$ (affine dots). The potential $W(l) = L^2/(l^2 + a^2)$ has its maximum at the throat $l = 0$, so the unstable circular orbit sits at the throat with $b_c = L/E = a$. Perturbing, $\delta\ddot{l} = -\frac{1}{2}W''(0)\delta l$ with $W''(0) = -2L^2/a^4$, hence $\delta\ddot{l} = (L^2/a^4)\delta l$ and the affine exponent is $\lambda_{\text{aff}} = L/a^2$. On the orbit $\dot{t} = E$ and $L = Ea$, so the coordinate-time exponent is $\lambda_t = \lambda_{\text{aff}}/E = 1/a$. The orbital frequency is $\Omega_c = \dot{\phi}/\dot{t} = (L/a^2)/E = 1/a$, so $\lambda_t/\Omega_c = 1$ for the Ellis throat, the same ratio as Schwarzschild.

A.3 Tangherlini photon spheres

Proposition A.1. *For the d -dimensional Tangherlini metric $f(r) = 1 - \mu r^{-(d-3)}$, the unstable circular photon orbit has $\lambda_t/\Omega_c = \sqrt{d-3}$.*

Proof. The photon-sphere condition $rf'(r) = 2f(r)$ reads $(d-3)\mu r^{-(d-3)} = 2f(r)$, since $f'(r) = (d-3)\mu r^{-(d-2)}$. Writing $u = \mu r^{-(d-3)}$ at the orbit, the condition is $(d-3)u = 2(1-u)$, so $u = 2/(d-1)$ and $f(r_c) = (d-3)/(d-1)$. Next, $f''(r) = -(d-3)(d-2)\mu r^{-d}$, so $r_c^2 f''(r_c) = -(d-3)(d-2)u = -2(d-3)(d-2)/(d-1)$. Insert into (3) with $\Omega_c^2 = f(r_c)/r_c^2$:

$$\lambda_t^2 = \frac{\Omega_c^2}{2} \left[2f(r_c) - r_c^2 f''(r_c) \right] = \frac{\Omega_c^2}{2} \cdot \frac{2(d-3)}{d-1} [1 + (d-2)] = (d-3)\Omega_c^2,$$

since $2f(r_c) = 2(d-3)/(d-1)$. Hence $\lambda_t/\Omega_c = \sqrt{d-3}$. $\square$

For $d = 4$ this recovers $\lambda_t = \Omega_c = 1/(3\sqrt{3}M)$ at $\mu = 2M$. The proposition is a closed-form special case of the general instability analysis of (Cardoso et al., 2009); we have not found the ratio stated in this form in the literature, and flag it accordingly in Section 6.

B Numerical methods and data provenance

All numerical work is Python (numpy, scipy) with $G = c = 1$. Code, data, and figures are archived with the paper’s workspace; every quantitative claim in the main text traces to a row of Table 4.

Single black holes (`code/exp_bh.py`). Schwarzschild exponents by (i) the closed form (3); (ii) paired-ray deviation growth from $\delta r_0 = 10^{-6}$ at $r = 3$, fitted over $3 \times 10^{-6} < |\delta r| < 3 \times 10^{-2}$ ($R^2 = 0.9999995$); (iii) fits to the winding and dwell-time divergences over 140 impact parameters in (4.9, 5.6) refined to $|b - b_c| = 10^{-4}$ (77 escaped, 63 captured, no timeouts). The whole-trajectory FTLE uses a paired ray at $\delta b = 10^{-7}$. Kerr exponents use the radial-potential curvature (R'') method for equatorial circular photon orbits, with circular-condition residuals below 2×10^{-13} and the $a = 0$ row reproducing Schwarzschild exactly; the $a = 0.9$ near-critical check uses 12 scattering rays and log-slope fits, not a paired-ray FTLE. Fisher bounds use analytic derivatives of the point-lens image positions and time delay with the noise normalizations stated in the text.

Di-hole (`code/exp_dihole.py`, `code/exp_zoom_extra.py`). Batched fixed-step RK4 in the optical parametrization ($h_t = 0.05$); constraint drift $\leq 3.4 \times 10^{-8}$ (max, production runs). The affine parametrization was rejected after a step study showed $O(1)$ drift and outcome misclassification at fixed step. Capture radius 0.08, escape radius 80, $T_{\max} = 350$. The uncertainty exponent is measured two ways: grid-window census over $\epsilon \in [0.004, 0.128]$ (primary) and perturbation probes on 500 base points (consistency bound; counting floor 1/500). The D5 coarse-graining measure: for each cell width, each fine-grid outcome is predicted by the majority outcome of its cell; accuracy and mean within-cell Shannon entropy are recorded.

Wormhole, dimensions, media (`code/exp_media.py`, `code/fig_media.py`). Ellis exponents by the closed form of Appendix A and by deviation integration at $\delta l = 10^{-6}$; scattering by batched RK4 over 92 impact parameters. Tangherlini values are closed-form (Proposition A.1) with a $d = 5$ deviation-integration check. Plasma rays by batched RK4 in the Hamiltonian formulation $H = \frac{1}{2}(|\mathbf{p}|^2 - n^2(\mathbf{x}))$; the sampled minimum of n^2 is 0.494 at $A = 0.5$, and no ray entered an $n^2 \leq 0$ region. The plasma uncertainty exponent uses the same probe protocol as the di-hole census for comparability.

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Table 4. Data provenance: experiment, code, data, figures.

Run	Code	Data	Figures
E1–E4 (BH)	<code>exp_bh.py</code>	<code>bh_results.npz</code>	<code>fig_bh_*.png</code>
D1–D5 (di-hole)	<code>exp_dihole.py</code>	<code>dihole_results.npz</code>	<code>fig_dihole_*.png</code>
W1–W2, P1 (media)	<code>exp_media.py</code>	<code>media_results.npz</code>	<code>fig_media_*.png</code>
P1 scan	<code>exp_plasma_scan.py</code>	<code>plasma_scan_results.npz</code>	<code>fig_media_plasma_scan.png</code>

Notes: Headline numbers are collected in `bh_summary.json`, `dihole_summary.json`, and `media_summary.json`; methods and failure logs in the companion memos in `data/`. All figures were produced with the shared chart style and passed its QA check.

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